

ON DEL PEZZO ELLIPTIC VARIETIES OF DEGREE ≤ 4

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ABSTRACT. Let Y be a del Pezzo variety of degree $d \leq 4$ and dimension $n \geq 3$, let H be an ample class such that $-K_Y = (n-1)H$ and let $Z \subset Y$ be a 0-dimensional subscheme of length d such that the subsystem of elements of $|H|$ with base locus Z gives a rational morphism $\pi_Z: Y \dashrightarrow \mathbb{P}^{n-1}$. Denote by $\pi: X \rightarrow \mathbb{P}^{n-1}$ the elliptic fibration obtained by resolving the indeterminacy locus of π_Z . Extending the results of [5] we study the geometry of the variety X and we prove that the Mordell-Weil group of π is finite if and only if the Cox ring of X is finitely generated.

INTRODUCTION

Let Y be a del Pezzo variety of dimension $n \geq 3$ and H an ample class such that $-K_Y = (n-1)H$ and let $d := H^n$ be the degree of Y . We consider the rational map $\pi_Z: Y \dashrightarrow \mathbb{P}^{n-1}$ associated to a linear series $V \subset |H|$ of dimension $n-1$, having 0-dimensional base locus Z . In what follows we say that the map $\pi: X \rightarrow \mathbb{P}^{n-1}$, obtained by resolving the indeterminacy of π_Z , is a *del Pezzo elliptic fibration* while X is a *del Pezzo elliptic variety* of degree d .

In [3] the case of general V is considered in relation with the Morrison-Kawamata cone conjecture. In [5] the case $\deg(Y) = 3$ has been studied, providing the Mordell-Weil groups of all the types of fibrations that can be obtained and proving that the group is finite if and only if the Cox ring of X is finitely generated.

In this paper we extend the results of [5] to del Pezzo elliptic varieties of degree ≤ 4 . Our first result is about the Mordell-Weil groups of the corresponding del Pezzo elliptic fibrations (the notation will be explained in Section 2).

Theorem 1. *The Mordell-Weil groups of the del Pezzo elliptic fibrations of degree $d \leq 4$ and dimension $n \geq 3$ are the following:*

Degree	Type	MW(π)	Degree	Type	MW(π)
1	X_1	$\langle 0 \rangle$	3	X_3, X_S	$\langle 0 \rangle$
2	X_{11}	\mathbb{Z}	4	X_{40}	\mathbb{Z}^3
	X_{SS}	$\mathbb{Z}/2\mathbb{Z}$		X_{41}, X_{30}	\mathbb{Z}^2
	X_2	$\langle 0 \rangle$		X_{42}	$\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})$
3	X_{111}	\mathbb{Z}^2		X_{31}, X_{20}, X_{21}	\mathbb{Z}
	X_{S11}, X_{12}	\mathbb{Z}		X_{43}	$(\mathbb{Z}/2\mathbb{Z})^2$
	X_{SSS}	$\mathbb{Z}/3\mathbb{Z}$		X_{21}, X_{22}	$\mathbb{Z}/2\mathbb{Z}$
	X_{S2}	$\mathbb{Z}/2\mathbb{Z}$		X_{10}, X_{11}	$\langle 0 \rangle$

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Table 1: Mordell-Weil groups of del Pezzo elliptic fibrations

Our second result is about Cox rings of elliptic del Pezzo varieties.

Theorem 2. *Let X be a del Pezzo elliptic variety of degree ≤ 4 and dimension $n \geq 3$. Then the following are equivalent:*

- (1) *the Cox ring of X is finitely generated;*
- (2) *the Mordell-Weil group of $\pi: X \rightarrow \mathbb{P}^{n-1}$ is finite.*

We prove Theorem 2 showing that any del Pezzo elliptic variety, whose corresponding elliptic fibration has finite Mordell-Weil group, is a Mori Dream Space and viceversa. Then we conclude by means of [6, Proposition 2.9]. The proof of our second theorem makes use of a detailed study of the structure of the moving and effective cones of elliptic del Pezzo varieties. In particular we prove the following.

Theorem 3. *Let $\pi: X \rightarrow \mathbb{P}^{n-1}$ be a del Pezzo elliptic fibration of degree $d \leq 4$ and having finite Mordell-Weil group. Then the effective cone $\text{Eff}(X)$ is generated by the vertical classes and the classes of sections, the cone $\text{Mov}(X)$ is the dual of $\text{Eff}(X)$ with respect to the bilinear form introduced in (1.2). The intersection graphs for the effective cones are given in the following table, where each vertex corresponds to a section or a vertical class D , the label in the vertex is $-\langle D, D \rangle$ and the number of edges connecting two vertices D and D' is $\langle D, D' \rangle$.*




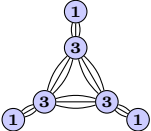
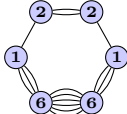
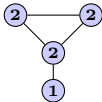
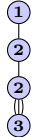
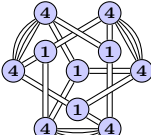
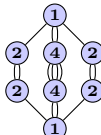
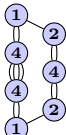
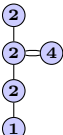
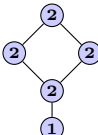
deg = 1	X_1 				
deg = 2	X_{SS} 	X_2 			
deg = 3	X_{SSS} 	X_{S^2} 	X_3 	X_S 	
deg = 4	X_{43} 	X_{21} 	X_{22} 	X_{11} 	X_{10} 

TABLE 2. Intersection graphs for the effective cones.

The paper is structured as follows. In Section 1, we introduce del Pezzo elliptic fibrations and del Pezzo elliptic varieties and we define the bilinear form on the Picard group of such varieties. In Section 2, we study the geometry of these varieties and in the next section we use these results in order to classify the Mordell-Weil groups of the del Pezzo elliptic fibrations, their vertical classes and sections. Section 4 contains the description of the nef, effective and moving cones of del Pezzo elliptic varieties and moreover in the same section we prove Theorem 2. In the last section, we provide the Cox rings of the del Pezzo elliptic varieties whose fibration has finite Mordell-Weil group and having degree one, two and four (few examples), and a lemma about the Cox ring of the blow-up in one point of the complete intersection of two quadrics.

1. DEL PEZZO ELLIPTIC VARIETIES

Let Y be a del Pezzo variety of dimension $n \geq 3$ such that $-K_Y = (n-1)H$, with H ample and $d := H^n \leq 4$. It is well known (see for instance [2]) that the Picard group of Y has rank one and it is generated by the class H . Let us recall the following. If $d = 1$ then Y is a smooth hypersurface of degree six of the weighted projective space $\mathbb{P}(3, 2, 1, \dots, 1)$ and H is the restriction of a degree one class of the ambient space. If $d = 2$ then Y is a double cover of \mathbb{P}^n branched along a smooth quartic hypersurface and H is the pull-back of a hyperplane of \mathbb{P}^n . If $d \in \{3, 4\}$ then Y is a projectively normal subvariety of \mathbb{P}^{n+d-2} and H is the class of a hyperplane section.

Let us consider a $n-1$ -dimensional sublinear system of $|H|$, whose base locus Z has dimension zero and length d . In particular, if $d = 1$ we have $Z = V(x_3, \dots, x_{n+2})$, if $d = 2$, Z is preserved by the covering involution and if $d \in \{3, 4\}$, Z spans a linear subspace $\Lambda \subseteq \mathbb{P}^{n+d-2}$ of dimension $d-2$. Let us denote by $\pi_Z: Y \dashrightarrow \mathbb{P}^{n-1}$ the rational map defined by the given system and by $\pi: X \rightarrow \mathbb{P}^{n-1}$ the resolution of the indeterminacy of π_Z . The variety X comes with two morphisms:

$$\begin{array}{ccc} X & \xrightarrow{\pi} & \mathbb{P}^{n-1} \\ \sigma \downarrow & \nearrow \pi_Z & \\ Y & & \end{array}$$

where σ is the composition of d blowing-ups $\sigma_1, \dots, \sigma_d$ at the points q_1, \dots, q_d , respectively. Moreover, assuming that Λ is not contained in the tangent space of Y at any point of Z when $d = 4$, the general fiber of π is a smooth genus one curve, that is π is an elliptic fibration.

In what follows, by abuse of notation, we use the same letter H to denote the pull-back of H via σ while we denote by E_i the pull-back of the exceptional divisor of σ_i , for $i \in \{1, \dots, d\}$. Observe that some of the points q_2, \dots, q_d can lie on the exceptional divisor of one of the σ_i 's. Therefore E_i can be either a \mathbb{P}^{n-1} or the union of a \mathbb{P}^{n-1} with some other components isomorphic to the projectivization \mathbb{F} of the vector bundle $\mathcal{O}_{\mathbb{P}^{n-1}} \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(1)$. In any case, we can write

$$\text{Pic}(X) = \langle H, E_1, \dots, E_d \rangle,$$

where, with abuse of notation, we are adopting the same symbols for the divisors and for their classes. We will also adopt the following notation

$$(1.1) \quad F := -\frac{1}{n-1}K_X.$$

Observe that F is the pull-back of a hyperplane section of \mathbb{P}^{n-1} via π , so that $F = H - \sum_{i=1}^d E_i$.

Remark 1.1. The map σ is a composition of blow-ups at points and we claim that we blow up at most one point on each exceptional divisor. Indeed, assume by contradiction that there is a prime divisor E which is the strict transform of an exceptional divisor blown up at two or more points. The preimage S of a general line ℓ of \mathbb{P}^{n-1} via π is a rational elliptic surface with nef anticanonical class which contains a prime divisor $E|_S$ of self-intersection < -2 , a contradiction.

1.1. A bilinear form on the Picard group. Let now X be the blow-up of Y at r general points. Using the above notation for F , we introduce a bilinear form on $\text{Pic}(X)$ by setting

$$(1.2) \quad \langle A, B \rangle := F^{n-2} \cdot A \cdot B$$

for any two divisors A and B on X . Thus the quadratic form q induced by the above linear form is hyperbolic and the matrix with respect to the basis (H, E_1, \dots, E_r) is diagonal with entries $d, -1, \dots, -1$. Since $\langle F, F \rangle = d - r$, the sublattice F^\perp is negative definite if $1 < r < d$ and it is negative semidefinite if $r = d$. In the first case, a basis consists of the classes $E_1 - E_2, \dots, E_{r-1} - E_r$, while in the second case it consists of the above classes plus F . These are roots lattices of type A_{r-1} and \tilde{A}_{d-1} , respectively.

When $r = d$ and the linear system $|F|$ on the blow-up X induces the elliptic fibration $\pi: X \rightarrow \mathbb{P}^{n-1}$, we observe that F^{n-2} is rationally equivalent to a smooth rational elliptic surface S which is the preimage via π of a line. Thus we have $\langle A, B \rangle = A|_S \cdot B|_S$, where the right hand side is the intersection product in $\text{Pic}(S)$.

Proposition 1.2. *Let A and B be effective divisors of X with B a prime divisor. If $\langle A, B \rangle < 0$ then B is contained in the stable base locus of $|A|$.*

Proof. Let ℓ be a general line of \mathbb{P}^{n-1} and let S be the surface $\pi^{-1}(\ell)$. According to the definition of the bilinear form we have $A|_S \cdot B|_S < 0$. Being B prime and ℓ general, the divisor $B|_S$ of S is prime as well. Thus the linear series $|A|_S|$ contains $B|_S$ into its base locus and the same holds for the linear series $|A|$. Varying ℓ we get the claim. \square

2. TYPES

In this section we are going to describe the possible types of del Pezzo elliptic varieties of degree $d \leq 4$.

2.1. Degree one. In this case Y is a degree 6 hypersurface of the $(n+1)$ -dimensional weighted projective space $\mathbb{P}(3, 2, 1, \dots, 1)$. After applying a change of coordinates, we can assume that a defining equation for Y is

$$(2.1) \quad x_1^2 - x_2^3 + x_2 f_4 + f_6 = 0,$$

where f_t is a degree t homogeneous polynomial in x_3, \dots, x_{n+2} . The blow-up $\sigma: X \rightarrow Y$ is centered at the point $q = (1, 1, 0, \dots, 0) \in Y$ and the rational map $Y \dashrightarrow \mathbb{P}^{n-1}$ is defined by $(x_1, \dots, x_{n+2}) \mapsto (x_3, \dots, x_{n+2})$.

2.2. Degree two. In this case Y is a double covering $\varphi: Y \rightarrow \mathbb{P}^n$ branched along a smooth quartic hypersurface S . In order to distinguish the different cases that can occur, we observe that the preimage of a line ℓ through a point $p := \varphi(q_1)$ is one of the following:

$$(2.2) \quad \varphi^{-1}(\ell) = \begin{cases} \text{elliptic curve} & \text{if } |\ell \cap S| = 4 \\ \text{rational nodal curve} & \text{if } |\ell \cap S| = 3 \\ \text{union of two smooth rational curves} & \text{if } \ell \text{ is bitangent to } S. \end{cases}$$

Therefore we distinguish three different cases depending on the position of p with respect to S and on the dimension of the variety $B \subseteq \mathbb{P}^n$ spanned by the bitangent to S passing through p .

Case 1. The point p does not lie on S and B is not a hypersurface. In this case the preimage of p in the double covering $Y \rightarrow \mathbb{P}^n$ consists of two distinct points q_1 and q_2 . We denote by X_{11} the variety that we obtain by blowing up these two points.

Case 2. The point p does not lie on S and B is a hypersurface. In this case after a linear change of coordinates we can assume that $p = (0, \dots, 0, 1)$. An equation for Y has the following form

$$(2.3) \quad x_{n+2}^2 = g + h^2,$$

where $g \in \mathbb{C}[x_1, \dots, x_n]$ is a homogeneous polynomial of degree four such that $V(g)$ is the cone spanned by the bitangents through p , while $h \in \mathbb{C}[x_1, \dots, x_{n+1}]$ is a homogeneous polynomial of degree two such that $h(p) \neq 0$ and $S = V(g + h^2)$. We denote by X_{SS} the variety obtained by blowing up the two distinct points q_1 and q_2 in the pre-image of p .

Case 3. The point p lies on S . In this situation B cannot be a hypersurface since otherwise S would be singular. In order to get an elliptic fibration we need to blow up the point $q_1 := \varphi^{-1}(p)$ and the point on the exceptional divisor which is invariant with respect to the lifted involution. We denote by X_2 the variety that we obtain after the blowing-ups. In this case an equation for Y has the following form

$$(2.4) \quad x_{n+2}^2 = x_n x_{n+1}^3 + f.$$

where $f \in \mathbb{C}[x_1, \dots, x_{n+1}]$ is a homogeneous polynomial of degree four which does not contain monomials of degree ≥ 3 in the variable x_{n+1} . The point q_1 has coordinates $(0, \dots, 0, 1, 0)$ and the tangent space to S at p is $V(x_n)$.

2.3. Degree four. Let us first collect some fact about smooth complete intersections of two hyperquadrics $Y := Q \cap Q' \subseteq \mathbb{P}^{n+2}$, for $n \geq 3$. Observe that any quadric in the pencil Λ generated by Q and Q' has rank at least $n+2$, since otherwise Y would not be smooth, and there are $n+3$ singular quadrics in the pencil, counting multiplicities. We claim that there are exactly $n+3$ quadrics of rank $n+2$ and their vertices are in general position in \mathbb{P}^{n+2} . Indeed, let us suppose that either there are less than $n+3$ vertices or that they are not in general position.

In the former case the pencil of quadrics is tangent to the discriminant hypersurface at some point. Without loss of generality we can assume Q to be a cone of

vertex $p = (1, 0, \dots, 0)$ in diagonal form g and the pencil $g + tg'$ is tangent to the discriminant hypersurface at $t = 0$. If the Hessian matrix of g is M and that of g' is M' , the above tangency condition is equivalent to the vanishing of the following derivative

$$\frac{d}{dt} \text{Det}(M + tM')|_{t=0}.$$

Expanding the above derivative and using the fact that M is diagonal we see that the above is equivalent to $m'_{11} = 0$, that is $p \in Q'$. This is not possible since it contradicts the smoothness of Y .

In the latter case there exists a hyperplane $H \subseteq \mathbb{P}^{n+2}$ containing all the vertices and if we restrict Λ to H we obtain a pencil Λ_H of quadrics in \mathbb{P}^{n+1} , containing at least $n+3$ singular quadrics (counting multiplicities). Hence all the quadrics of Λ_H must be singular and by Bertini's theorem their vertices are contained in the base locus of Λ_H . This implies that all the vertices of these cones are in Y and this is a contradiction since they give singular points of Y .

In what follows we will denote by Q_1, \dots, Q_{n+3} the singular quadrics and by p_1, \dots, p_{n+3} the corresponding vertices. By the above discussion, we can assume that p_i is the i -th fundamental point of \mathbb{P}^{n+2} for $i = 1, \dots, n+3$, so that Q_1 and Q_2 are defined by diagonal forms. Moreover, after possibly rescaling the variables, we can assume the quadrics to be defined by the following polynomials

$$(2.5) \quad x_2^2 - x_3^2 + x_4^2 + \dots + x_{n+3}^2 \quad x_1^2 - x_3^2 + \alpha_4 x_4^2 + \dots + \alpha_{n+3} x_{n+3}^2$$

respectively, where the coefficients α_i are distinct and not in $\{0, 1\}$. Let us now prove the following result that will be useful in the next section.

Proposition 2.1. *Let q_1 and q_2 be two points of Y , possibly infinitely closed. Then the conics of Y through these two points span a hypersurface of Y if and only if the line $\langle q_1, q_2 \rangle$ passes through one of the vertices p_i .*

Proof. If p_i lies on the line $\langle q_1, q_2 \rangle$, then this line is a generatrix of the cone Q_i . We can write $Y = Q_i \cap Q$, where Q is any other quadric of the pencil. We conclude observing that there exists an $(n-2)$ -dimensional family of planes of Q_i containing a generatrix and each of them intersects Q along a conic through the two fixed points.

Let us suppose now that the conics through q_1 and q_2 span a hypersurface S , i.e. there exists an $(n-2)$ -dimensional family of such conics. Observe that when we have a conic C contained in Y then the plane π_C of the conic must be contained in one of the quadrics of the pencil Λ (since the generic quadric of the pencil cuts π_C along the curve C , imposing to pass through one point of π_C not lying on C we get the whole plane). Therefore, under our hypotheses, we must have a quadric of the pencil containing an $(n-2)$ -dimensional family of planes sharing the line $\langle q_1, q_2 \rangle$. Hence this quadric is a cone with vertex on that line. \square

Let us fix now a plane $\Pi \subseteq \mathbb{P}^{n+2}$ and let us analyze the different types of del Pezzo elliptic varieties of degree four. By Proposition 2.1 the type depends not only on the number of points we blow up but also on the number of vertices p_i contained in the plane Π . Hence we are going to use the symbol X_{kl} to denote the variety that we obtain by choosing a plane Π intersecting Y in k distinct points and containing l vertices.

Let us spend few words about the geometry of this construction and about the possible values of k and l for X_{kl} . We remark that we can write

$$\Pi \cap Y = C \cap C'$$

where $C := Q \cap \Pi$ and $C' := Q' \cap \Pi$ are two plane conics. We discuss four cases.

Case 1. If Π contains no vertices, then we have two smooth conics, whose intersection consists of k distinct points, for $k \in \{1, 2, 3, 4\}$ and hence we get the types X_{40} , X_{30} , X_{20} , X_{10} . Observe that when $k = 2$ we have two possibilities: either C and C' are tangent at their two intersection points q_1 and q_2 , or they intersect transversally at q_2 and with multiplicity three at q_1 .

Case 2. If Π contains one vertex p_i , then we can suppose that C is a smooth conic while $C' := \Pi \cap Q_i$ has (at least) a singular point at the vertex $p_i \in \Pi$. The intersection of C and C' consists of k points, for $k \in \{1, 2, 3, 4\}$ and we obtain the types X_{41} , X_{31} , X_{21} , X_{11} . As before, when $k = 2$ we have two possibilities. Either C' is the union of two distinct lines and each of them is tangent to the conic C , or C' is a double line (which means that Π is tangent to Q_i) intersecting C in two distinct points.

Case 3. If Π contains two vertices, then we can suppose that both C and C' are singular and they can not intersect at the vertices so that k can be either 1, 2 or 4. Moreover, when $k = 1$ we deduce that the plane Π is contained in the tangent space to Y at the only intersection point q_1 . We are not going to consider this case since it does not give an elliptic fibration, being all the fibers singular rational curves. Hence we have only the two types X_{42} and X_{22} .

Case 4. Finally, observe that if Π contains three vertices then it is fixed and it can intersect Y only at four distinct points (otherwise Y would be singular), giving case X_{43} .

Remark 2.2. We provide here an example of defining equations for Π for each of the following five types:

$$X_{43} : \quad \Pi = V(x_4, x_5, \dots, x_{n+3})$$

$$X_{22} : \quad \Pi = V(x_3 - x_4, x_5, \dots, x_{n+3})$$

$$X_{21} : \quad \Pi = V(\sqrt{\alpha_4 + \alpha_5} \cdot x_2 - \sqrt{\alpha_4 + \alpha_5 - 2} \cdot x_3, x_4 - x_5, \dots, x_{n+3})$$

$$X_{11} : \quad \Pi = V(x_1 - \alpha_4 x_2 + (\alpha_4 - 1)x_3, x_5, \dots, x_{n+3})$$

$$X_{10} : \quad \Pi = V(2x_1 - (\alpha_4 + \alpha_5)x_2 + (\alpha_4 + \alpha_5 - 2)x_3, x_4 - x_5, x_6, \dots, x_{n+3}),$$

where $\alpha_4 + \alpha_5 \neq 0, 2$ in cases X_{21} and X_{10} .

Remark 2.3. We recall that if $Y = Q \cap Q' \subseteq \mathbb{P}^{n+2}$ and $n \geq 3$, then through any point of Y we have at least one line of Y . So let us fix a point $q_i \in Y$ and a line ℓ of Y , passing through this point, and let us describe the fiber of $\pi: X \rightarrow \mathbb{P}^{n-1}$ containing the strict transform of that line. The image of this fiber inside Y is the curve obtained by intersecting Y with the \mathbb{P}^3 spanned by the plane Π and the line ℓ . This can also be described as the base locus of the pencil of quadric surfaces obtained by restricting Λ to the \mathbb{P}^3 that we are considering. Observe that any time we have a vertex p_i in Π , the intersection of Q_i with the \mathbb{P}^3 is a quadric cone containing a line not passing through p_i . Hence it must be the union of two planes intersecting along a line passing through p_i . Therefore, in Case 1 the image of the fiber inside Y is obtained by intersecting two smooth quadric surfaces sharing a line and hence it is the union of that line and a rational normal cubic, intersecting

in two points. In Case 2 the base locus is the intersection of a smooth quadric with a reducible one and then it is the union of two lines and a smooth conic. In Case 3 the base locus is the intersection of two reducible quadrics and hence it consists of four lines. Finally, in Case 4 we have the base locus of a pencil containing three reducible quadrics. Thus, after a possible renaming of the coordinates, the pencil has the form $(x_1^2 - x_2^2) + t(x_2^2 - x_3^2)$. All the quadrics in this pencil are singular at the point $p = (1, 1, 1, 1)$ and the base locus of the pencil consists of four lines intersecting at the point p . We remark that in this last case the corresponding fiber in X is the union of four rational components passing through one point and hence it is a type that does not appear in the Kodaira's list of singular fibers for elliptic surfaces.

3. MORDELL-WEIL GROUPS

The main result of this section is the proof of Theorem 1 but we postpone it to the end of the section and we begin by studying the *vertical divisors* of all the del Pezzo elliptic fibrations of degree $d \leq 4$, that is divisors D such that $\pi(D)$ is a hypersurface of \mathbb{P}^{n-1} . If $d = 1$, then the only vertical class is F since the rank of the subgroup of vertical divisors equals $\text{rk Pic}(X) - 1 = 1$.

When $d = 2$, recall that there is a double covering $\varphi: Y \rightarrow \mathbb{P}^n$ branched along a smooth quartic hypersurface S and $\sigma: X \rightarrow Y$ is the blow-up of Y at two points q_1, q_2 exchanged by the covering involution. By (2.2), if D is a prime proper vertical divisor of X whose class is not a multiple of F , then either D is contained in the pull-back of an exceptional divisor of σ , or $\varphi(D)$ is covered by bitangent lines to S . Therefore in case X_{11} there are no proper vertical divisors.

In case X_{SS} we have the two vertical classes $2H - 4E_1, 2H - 4E_2$, and assuming that Y has the equation (2.3), they are the classes of the strict transforms of $V(x_{n+2} - h)$ and $V(x_{n+2} + h)$, respectively.

Finally, in case X_2 , $E_1 - E_2$ and $H - 2E_1$ are the only prime proper vertical divisors.

The case $d = 3$ has already been studied in [5] and we refer to that paper for the classification of prime proper vertical divisors.

For $d = 4$, if D is a prime proper vertical divisor, then D is strictly contained in the support of $\pi^*\pi(D)$. Let γ be a general fiber of π over a point $q \in \pi(D)$ and let us denote by C the image $\sigma(\gamma)$ in Y . Then either (j) C is an irreducible rational curve or (jj) it contains lines and/or conics.

In case (j), C is singular at one of the points $q_i \in \Pi \cap Y$ and the union of these curves gives a prime proper divisor having class $H - 2E_i - E_j - E_k$. In order to obtain the class of a fiber we have to add some prime proper vertical exceptional divisors of the form $E_i - E_j$.

In case (jj), observe that by [3] through any point of Y there is only a $(n - 3)$ -dimensional family of lines and hence they can not fill up a divisor. Therefore the curve C must contain a conic through two points q_i and q_j of $\Pi \cap Y$, possibly infinitely near. By Proposition 2.1 the line $\langle q_i, q_j \rangle$ passes through one vertex p_k and hence $p_k \in \Pi$. In this case the class of one of the irreducible components of $\pi^*\pi(D)$ is of the form $H - 2E_i - 2E_j$. For instance, in case X_{31} we can write $Y = Q_1 \cap Q$, where Q_1 is a cone with vertex $p_1 \in \Pi$ and Q is a smooth quadric. Furthermore, Q intersects Π along a smooth conic C while $Q_1 \cap \Pi$ is the union of two generatrices and one of them is tangent to C at q_1 while the other one intersects

C in q_3 and q_4 . Therefore we have the vertical class $H - 2E_1 - 2E_2$ corresponding to the conics through q_1 and whose tangent line at q_1 is the line $\langle q_1, p_1 \rangle$ and the class $H - 2E_3 - 2E_4$ corresponding to the conics through q_3 and q_4 . Observe that the sum of these classes gives twice a fiber. Moreover, we also have the vertical class $E_1 - E_2$ sitting inside the exceptional locus and the class $H - 2E_1 - E_3 - E_4$ which is spanned by the union of the strict transforms of the singular rational quartic curves of Y obtained intersecting it with a hyperplane tangent to Y at q_1 .

We summarise the above observations in the following

Proposition 3.1. *Let $\pi: X \rightarrow \mathbb{P}^{n-1}$ be a del Pezzo elliptic fibration of degree $d \leq 4$ and dimension $n \geq 3$. Then for each type the sections and the vertical divisors are as follows:*

Degree	Type	Sections	Proper prime vertical divisors
1	X_1	E_1	
2	X_{11} X_{SS} X_2	E_1, E_2 E_1, E_2 E_2	$2H - 4E_1, 2H - 4E_2$ $E_1 - E_2, H - 2E_1$
3	X_{111} X_{S11} X_{SSS} X_{12} X_{S2} X_3 X_S	E_1, E_2, E_3 E_1, E_2, E_3 E_1, E_2, E_3 E_1, E_3 E_1, E_3 E_3 E_3	$H - 3E_1, 2H - 3E_2 - 3E_3$ $H - 3E_1, H - 3E_2, H - 3E_3$ $E_2 - E_3, H - E_1 - 2E_2$ $H - 3E_1, 2H - 3E_2 - 3E_3, E_2 - E_3, H - E_1 - 2E_2$ $E_1 - E_2, E_2 - E_3, H - 2E_1 - E_2$ $E_1 - E_2, E_2 - E_3, H - 2E_1 - E_2, H - 3E_1, 2H - 3E_2 - 3E_3$
4	X_{40} X_{41} X_{42} X_{43} X_{30} X_{31} X_{20} X_{21} X_{22} X_{10} X_{11}	E_1, E_2, E_3, E_4 E_1, E_2, E_3, E_4 E_1, E_2, E_3, E_4 E_1, E_2, E_3, E_4 E_2, E_3, E_4 E_2, E_3, E_4 E_3, E_4 E_3, E_4 E_3, E_4 E_3, E_4 E_3, E_4 E_4 E_4	$H - 2E_1 - 2E_2, H - 2E_3 - 2E_4$ $H - 2E_1 - 2E_2, H - 2E_3 - 2E_4$ $H - 2E_1 - 2E_3, H - 2E_2 - 2E_4$ $H - 2E_i - 2E_j, 1 \leq i < j \leq 4$ $E_1 - E_2, H - 2E_1 - E_3 - E_4$ $H - 2E_1 - 2E_2, H - 2E_3 - 2E_4$ $E_1 - E_2, H - 2E_1 - E_3 - E_4$ $E_1 - E_3, H - 2E_1 - E_2 - E_4, E_2 - E_4, H - E_1 - 2E_2 - E_3$ $E_1 - E_3, E_3 - E_4, H - 2E_1 - E_2 - E_3$ $E_1 - E_3, E_2 - E_4, H - 2E_1 - 2E_3, H - 2E_2 - 2E_4$ $H - 2E_1 - E_2 - E_4, H - E_1 - 2E_2 - E_3$ $E_1 - E_3, E_2 - E_4, H - 2E_1 - E_2 - E_4, H - E_1 - 2E_2 - E_3$ $H - 2E_1 - 2E_3, H - 2E_2 - 2E_4$ $E_1 - E_3, E_2 - E_4, H - 2E_1 - 2E_2$ $E_1 - E_2, E_2 - E_3, E_3 - E_4, H - 2E_1 - E_2 - E_3$ $E_1 - E_2, E_2 - E_3, E_3 - E_4, H - 2E_1 - 2E_2$

Table 3: Sections and vertical classes of del Pezzo elliptic fibrations with $d \leq 4$.

Proof of Theorem 1. Recall that the Mordell-Weil group of the elliptic fibration π is the group of rational sections of π or, equivalently, the group of $K = \mathbb{C}(\mathbb{P}^{n-1})$ -rational points $X_\eta(K)$ of the generic fiber X_η of π once we choose one of such points O as an origin for the group law. Let \mathcal{T} be the subgroup of $\text{Pic}(X)$ generated by the classes of the vertical divisors and by the class of the section O . There is an

exact sequence [7, Section 3.3]:

$$(3.1) \quad 0 \longrightarrow \mathcal{T} \longrightarrow \mathrm{Pic}(X) \longrightarrow X_\eta(K) \longrightarrow 0.$$

In degree d , the Picard group of X is free of rank $d + 1$, generated by the classes H, E_1, \dots, E_d . Observe that if F is defined as in (1.1), then $\langle F, E_d \rangle \subseteq \mathcal{T}$ holds and by Proposition 3.1 and the sequence (3.1) we get the statement. \square

4. CONES

The aim of this section is to provide a description of the nef, effective and moving cone of del Pezzo elliptic varieties. Moreover, we discuss the Mori chamber decomposition of the moving cones and we use this decomposition in order to prove Theorem 2.

4.1. The nef cones. Given a subset I of $\{1, \dots, d\}$, in what follows we denote by F_I the divisor $H - \sum_{i \in I} E_i$.

Theorem 4.1. *Let $\pi: X \rightarrow \mathbb{P}^{n-1}$ be a del Pezzo elliptic fibration with $n \geq 3$. Then the extremal rays of the nef cone $\mathrm{Nef}(X)$ are all the F_I such that $I \subseteq \{1, \dots, d\}$ and $\langle F_I, V \rangle \geq 0$ for each exceptional vertical class V .*

Proof. Let us consider the subcone \mathcal{C} of the Mori cone of X generated by the following classes:

- e_i such that E_i is a section;
- $e_i - e_j$ such that $E_i - E_j$ is a prime vertical divisor;
- $h - e_i$ for each $q_i \in Y$.

Let $D := \alpha H - \sum m_i E_i$ be a class in the dual \mathcal{C}^* . Then we have the following inequalities: $m_i \geq 0 \forall i$, $m_i \geq m_j$ if the point q_j lies on the exceptional divisor of the blowing-up at q_i and finally $\alpha \geq m_i \forall i$. Let us write $\{m_1, \dots, m_d\} = \{\mu_1, \dots, \mu_r\}$, where $r \leq d$ and $0 = \mu_0 \leq \mu_1 < \dots < \mu_r$, and let us denote by $I_i := \{j \mid m_j \geq \mu_i\}$, for each $i = 1, \dots, r$. Then we can write

$$D = (\alpha - \mu_r)H + \sum_{i=1}^r (\mu_i - \mu_{i-1})F_{I_i},$$

where the F_{I_i} are nef and their product with any effective $E_j - E_k$ is non negative. In order to conclude the proof we need to show that these F_I are extremal rays of the nef cone.

Let us first suppose that X is obtained by blowing up d distinct points on Y . In this case, we have to consider all the F_I as I varies in the subsets of $\{1, \dots, d\}$, and by induction on d it can be proved that they are vertices of a d -dimensional hyper-cube. In particular, they are extremal rays of the cone they generate.

In addition, we can also infer that no F_I lies in the convex hull of the remaining and hence the general case follows. \square

4.2. The effective and moving cones. We now restrict our attention to del Pezzo elliptic fibrations of degree $d \leq 4$ and having finite Mordell-Weil group, proving Theorem 3.

Proof of Theorem 3. Let us consider, for each del Pezzo elliptic variety X the cone \mathcal{M} of $\text{Pic}_{\mathbb{Q}}(X)$ generated by the vertical classes and the sections of π . Let $\varrho_1, \dots, \varrho_n$ be the extremal rays of \mathcal{M} . We have the following inclusions

$$(4.1) \quad \bigcap_{i=1}^n \text{cone}(\varrho_1, \dots, \overset{\vee}{\varrho_i}, \dots, \varrho_n) \subseteq \text{Mov}(X) \subseteq \text{Eff}(X)^{\vee} \subseteq \mathcal{M}^{\vee},$$

where the first inclusion is due to the fact that each ρ_i is generated by a prime divisor, the second one is a consequence of Proposition 1.2 and the last one follows from $\mathcal{M} \subseteq \text{Eff}(X)$. The proof goes as follows. If the degree d is at most three, then the Cox ring is known (Theorem 5.1 for degree one or two and [5] for degree three) and a direct computation shows that the rays of \mathcal{M}^{\vee} are movable. When $d = 4$, observe that if X is of type X_{43}, X_{22}, X_{21} , then the first cone and the last one in (4.1) are equal and the two assertions of the theorem follow. In the remaining cases, we are going to check that the rays of \mathcal{M}^{\vee} are movable.

If X is of type X_{11} , then the only class we have to check is $H - 2E_1$ (all the other rays of \mathcal{M}^{\vee} are indeed nef classes). We are going to see that the base locus of the linear system $|H - 2E_1|$ has codimension two. Indeed, this linear system corresponds on Y to the linear system of hyperplane sections containing the tangent space at q_1 , whose base locus is the union of the lines passing through q_1 . When we blow up q_1 , the strict transforms of these lines intersect E_1 along a subvariety of codimension two. Observe that the second point q_2 that we blow up do not lie on this subvariety, since otherwise the plane Π would intersect Y along a line. Then the base locus of $|H - 2E_1|$ can not be divisorial.

In case X_{10} the only classes we have to check are $H - 2E_1$ and $3H - 4E_1 - 4E_2$. The first one can be done as in case X_{11} while the second one can be obtained as the image of H via the Geiser involution described in Subsection 4.3, and hence it is movable. \square

As a consequence of Theorem 3, if X is a del Pezzo elliptic variety of degree $d \leq 4$ such that the Mordell-Weil group of $\pi: X \rightarrow \mathbb{P}^{n-1}$ is finite, then the effective cone $\text{Eff}(X)$ can be read from Table 3. The graphs of the quadratic form on the primitive generators of the extremal rays of $\text{Eff}(X)$ are listed in Table 2.

Let us consider an example in which the Mordell-Weil group of the fibration is not finite and the moving cone is the union of infinitely many chambers.

When the elliptic fibration has degree $d = 2$ and type X_{11} , we have seen that the Mordell-Weil group is $\langle \sigma \rangle \cong \mathbb{Z}$. The action of σ on the Picard group of X , with respect to the basis $B := (H - E_1 - E_2, E_2 - E_1, E_1)$, is given by the following matrix

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

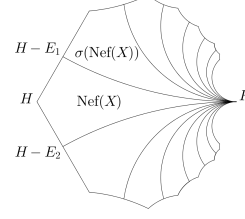
The cone $\sigma^k(\text{Nef}(X))$ is generated by the classes corresponding to the columns of the matrix

$$\begin{pmatrix} 1 & k^2 + k + 1 & k^2 - k + 1 & 2k^2 + 1 \\ 0 & k + 1 & k & 2k + 1 \\ 0 & 1 & 1 & 2 \end{pmatrix},$$

with respect to the basis B . We claim that the classes $\sigma^k(H)$ are extremal rays of the moving cone and generate it, so that the following equality holds

$$\text{Mov}(X) = \bigcup_{k \in \mathbb{Z}} \sigma^k(\text{Nef}(X)).$$

First of all, observe that for each $k \in \mathbb{Z}$ the cones $\sigma^k(\text{Nef}(X))$ and $\sigma^{k+1}(\text{Nef}(X))$ share the two-dimensional face generated by F and $\sigma^k(H - E_1) = \sigma^{k+1}(H - E_2)$. Moreover $\sigma^k(H) + \sigma^{k+1}(H) = 4\sigma^k(H - E_1)$, so that the union of the cones $\sigma^k(\text{Nef}(X))$ is a convex cone and the classes $\sigma^k(H - E_i)$, $i = 1, 2$, are on its boundary but they are not extremal rays. Now observe that the right hand side cone is contained in $\text{Mov}(X)$.



Finally, since the property of lying on the boundary of $\text{Mov}(X)$ is preserved by σ^k , we only have to prove that the two faces $\langle H, H - E_i \rangle$, for $i = 1, 2$, are on the boundary of the moving cone $\text{Mov}(X)$. We conclude observing that if we move outside from $\text{Nef}(X)$ along a direction orthogonal to the face $\langle H, H - E_1 \rangle$ (respectively $\langle H, H - E_2 \rangle$) we obtain classes containing E_2 (respectively E_1) in the stable base locus.

4.3. Generalized Bertini and Geiser involutions. We consider here a generalization of the classical Bertini and Geiser involutions to blow-ups of del Pezzo varieties. Let Y be a degree $d \geq 3$ del Pezzo variety and let $Z \subseteq Y$ be a zero-dimensional subscheme such that $\dim \langle Z \rangle = l(Z) - 1$ and the intersection of $d - 1$ general elements of $\mathcal{L}_Z := |\mathcal{O}_Y(1) \otimes \mathcal{I}_Z|$ is an elliptic curve. We denote by $\sigma: Y_Z \rightarrow Y$ the blow-up of Y along Z as in Section 1.

If Z has length $l(Z) = d - 2$, then the general $(d - 2)$ -dimensional linear space containing Z intersects $Y \setminus Z$ at two distinct points. The birational involution obtained by exchanging these two points induces a birational involution σ_G on the blow-up Y_Z of Y at Z . We call this σ_G a *generalized Geiser involution*.

When Z has length $l(Z) = d - 1$, denote by F the divisor on Y_Z defined as before. The base locus of the linear system $|F|$ consists of one point q while $|2F|$ defines a morphism φ . Since $F^n = 1$, we have that F^{n-1} is rationally equivalent to an elliptic curve C passing through q and the restriction $\varphi|_C$ is a double covering of a line passing through the point $p := \varphi(q)$. Hence the image $\varphi(Y_Z)$ is a cone V . If we denote by E the exceptional divisor corresponding to the last blow-up of σ , we have that the restriction $\varphi|_E$ is the 2-veronese embedding v_2 of \mathbb{P}^{n-1} . We conclude that V is a cone over $v_2(\mathbb{P}^{n-1})$ and φ induces a birational involution σ_B on Y_Z that we call a *generalized Bertini involution*. We remark that if X is the del Pezzo elliptic variety obtained by blowing up Y_Z in q , then σ_B induces on X the hyperelliptic involution with respect to the origin given by the exceptional divisor.

Remark 4.2. If Y has degree four and the line $\langle Z \rangle$ does not contain any vertex p_i , then the indeterminacy locus of the corresponding Geiser involution σ_G has codimension two. Moreover, it lifts to an isomorphism in codimension one for the elliptic varieties of type X_{21} and X_{10} . The action on the Picard group of X in each case is given by the following matrices respectively

$$\sigma_{21} = \begin{pmatrix} 3 & 1 & 1 & 0 & 0 \\ -4 & -2 & -1 & 0 & 0 \\ -4 & -1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad \sigma_{10} = \begin{pmatrix} 3 & 1 & 1 & 0 & 0 \\ -4 & -1 & -2 & 0 & 0 \\ -4 & -2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

To prove this, we first claim that the lifted birational map preserves the elliptic fibration π and thus it is a flop. Indeed, if f is a fibre of π whose image C in Y is cut out by a three-dimensional linear space L and we fix a point $y \in C$, then the plane spanned by y and $\langle Z \rangle$ is contained in L and thus it intersects C at a fourth point, so that $\phi(f) = f$, which proves the claim. Since ϕ preserves the fibration π , its pull-back ϕ^* must preserve both the sets of horizontal and vertical divisors of X . A direct calculation shows that the representative matrix for ϕ^* in the basis (H, E_1, \dots, E_4) is one of the above in each case.

4.4. Mori chambers. Let X be a del Pezzo elliptic variety of degree four with finite Mordell-Weil group. We provide here the Mori chamber decomposition of the moving cone $\text{Mov}(X)$ of X . In the following proposition, we will denote by N the nef cone of X and by

$$N_i := \text{cone}(\{F_I : i \in I \text{ and } F_I \in N\} \cup \{H - 2E_i\}).$$

Proposition 4.3. *Let X be a del Pezzo elliptic variety of degree four such that the corresponding elliptic fibration has finite Mordell-Weil group. Then the Mori chamber decomposition of $\text{Mov}(X)$ is given in the following table.*

Type	Cones
X_{43}	N, N_1, N_2, N_3, N_4
X_{22}	N, N_1, N_2
X_{21}	$N, N_1, N_2, \sigma_{21}^*(N), \sigma_{21}^*(N_1), \sigma_{21}^*(N_2)$
X_{11}	N, N_1
X_{10}	$N, N_1, \sigma_{10}^*(N), \sigma_{10}^*(N_1)$

Proof. Let $X \rightarrow X_i$ be the flop of the class $h - e_i$ of the strict transform C of a line through the point $q_i \in Y$. Note that such a flop exists by [3]. We show that the nef cone of X_i is N_i and then observe that the union of the cones in the table given in the statement is $\text{Mov}(X)$ for each type. To prove the claim, we begin by showing that the primitive generators of the extremal rays of the cone N_i are nef in X_i . Observe that each $F_I \in N_i$ is nef in both X and X_i since $F_I \cdot (h - e_i) = 0$ by our definition of N_i . Hence we only have to check that also $H - 2E_i$ is nef in X_i . Since $H - 2E_i$ is the pull-back of a class on the blow-up \tilde{Y} of Y at q_i , it is enough to prove the claim on \tilde{Y} . By Lemma 5.2 the Cox ring of \tilde{Y} is finitely generated and the moving cone decomposes as follows:

$$\text{Mov}(\tilde{Y}) = \text{cone}(H, H - E_i) \cup \text{cone}(H - E_i, H - 2E_i).$$

Thus, after flopping $h - e_i$ the class $H - 2E_i$ becomes nef as claimed so that we have the inclusion $N_i \subseteq \text{Nef}(X_i)$. To prove that this is indeed an equality, we show that the extremal rays of the dual cone of N_i are classes of effective curves of X_i . To this aim we make use of [3, Lemma 4.1] which asserts that if Γ is a curve of X which meets C transversally at k points, and no other effective curve of class $h - e_i$,

then the flop image Γ' of Γ has class

$$(4.2) \quad [\Gamma'] = [\Gamma] + k[C].$$

By a direct calculation, we see that the extremal rays of the dual cone of N_i are the following (here we list only the case $i = 1$, being the remaining cases analogous):

Type	Extremal rays of the Mori cone	Extremal rays of the dual cone of N_1
X_{43}	e_1, e_2, e_3, e_4 $h - e_1, h - e_2, h - e_3, h - e_4$	$-h + e_1, e_2, e_3, e_4,$ $2h - e_1 - e_2, 2h - e_1 - e_3, 2h - e_1 - e_4$
X_{22}, X_{21}	e_2, e_4 $h - e_1, h - e_3$ $e_1 - e_2, e_3 - e_4$	$-h + e_1, e_2, e_4$ $2h - e_1 - e_2, 2h - e_1 - e_3$ $e_3 - e_4$
X_{11}, X_{10}	$h - e_1, e_4, e_1 - e_2$ $e_2 - e_3, e_3 - e_4$	$-h + e_1, e_4, e_2 - e_3,$ $e_3 - e_4, 2h - e_1 - e_2$

For each type the curves having class e_i or $e_i - e_{i+1}$, with $i > 1$, do not intersect any curve of class $h - e_1$ and hence by (4.2) their classes in the Mori cone of X_1 are the same. Assume that Γ is an irreducible curve such that

$$[\Gamma] = 2h - e_1 - e_i.$$

We can assume that Γ is the strict transform of a smooth conic \mathcal{C} of Y passing through q_1 and q_i , which is possibly infinitely near to q_1 . The tangent line to \mathcal{C} at q_1 cannot be contained in Y since otherwise the plane spanned by \mathcal{C} and this line would be contained into each quadric of the pencil and thus in Y . This gives a contradiction, since the line through q_1 and q_i is not contained in Y by hypothesis. Thus we conclude again by (4.2), proving the assertion for X_{43}, X_{22} and X_{11} .

In case $X = X_{21}$ the chamber $\sigma_{21}^*(N)$ is the pull-back of the nef cone $N = \text{Nef}(X)$ via the flop σ_{21} . Since σ_{21} is the generator of the Mordell-Weil group of π , we deduce that $\sigma_{21}(X)$ is an elliptic del Pezzo variety of the same type. Thus each chamber $\sigma_{21}^*(N_i)$, for $i = 1, 2$, is a flop image of $\sigma_{21}^*(N)$ exactly as N_i is a flop image of N . In particular the chamber $\sigma_{21}^*(N_i)$ is generated by finitely many semiample classes of $\sigma_{21}(X_i)$.

Finally, in case $X = X_{10}$ we proceed as we did for X_{21} , considering σ_{10} instead of σ_{21} . □

Proof of Theorem 2. By [5, Lemma 3.5] (1) implies (2), so let us suppose that the Mordell-Weil group of π is finite. If $d = 1$ or 2 , then we conclude by means of Theorem 5.1, while the case $d = 3$ has been proved in [5, Theorem 3.6]. Finally, when $d = 4$, we observe that by Proposition 4.3, if the Mordell-Weil group of the fibration is finite, then the moving cone $\text{Mov}(X)$ satisfies all the hypotheses of [6]. □

5. COX RINGS

In this section, we provide a presentation for the Cox rings of the elliptic del Pezzo varieties of degree ≤ 4 . We recall that given a normal projective variety X

with finitely generated picard group, its Cox ring $\mathcal{R}(X)$ can be defined as (see [1])

$$\mathcal{R}(X) = \bigoplus_{[D] \in \text{Pic}(X)} H^0(X, \mathcal{O}_X(D)).$$

We apply [4, Algorithm 5.4] and we will explain all the steps in the algorithm for the convenience of the reader. Let Y_1 be a smooth projective variety with finitely generated Cox ring R_1 , which admits a presentation $R_1 = \mathbb{C}[T_1, \dots, T_{r_1}]/I_1$. Note that R_1 is K_1 -graded, where $K_1 = \text{Cl}(Y_1)$. Define $\bar{Y}_1 = \text{Spec}(R_1)$ and let $\hat{Y}_1 \subseteq \bar{Y}_1$ be the characteristic space of Y_1 with characteristic map $p: \hat{Y}_1 \rightarrow Y_1$. Let $q \in Y_1$ be the point that we want to blow up. We have the following commutative diagram:

$$\begin{array}{ccc} \overline{p^{-1}(q)} & \longrightarrow & \bar{Y}_1 \\ \uparrow & & \uparrow \\ p^{-1}(q) & \longrightarrow & \hat{Y}_1 \\ \downarrow p & & \downarrow p \\ q & \longrightarrow & Y_1. \end{array}$$

Let $I \subseteq R_1$ be the ideal of the closure of $p^{-1}(q)$ in \bar{Y}_1 , and let $J \subseteq R_1$ be the irrelevant ideal, i.e. the ideal of $\bar{Y}_1 \setminus \hat{Y}_1$. We choose a system of homogeneous generators $f_1, \dots, f_s \in R_1$ which form a basis for the ideal I and such that

$$(5.1) \quad f_i \in (I^{d_i} : J^\infty), \quad \forall i = 1, \dots, s,$$

where d_i is a positive integer. An ample class $[D]$ of Y_1 defines an embedding $Y_1 \rightarrow Z_1$ into a projective toric variety Z_1 , whose Cox ring is the K_1 -graded polynomial ring $\mathbb{C}[T_1, \dots, T_{r_1}]$ and such that the class $[D]$ is ample on Z_1 . We embed Z_1 into another toric variety W_1 via the following map

$$(T_1, \dots, T_{r_1}) \mapsto (T_1, \dots, T_{r_1}, f_1, \dots, f_s),$$

where the Cox ring of W_1 is the K_1 -graded polynomial ring $\mathbb{C}[T_1, \dots, T_{r_1+s}]$, with $\deg(T_{r_1+i}) = \deg(f_i)$ for any i , and again $[D]$ is an ample class of W_1 . Now we blow-up W_1 equivariantly along the orbit $V(T_{r_1+1}, \dots, T_{r_1+s})$, obtaining the toric variety Z_2 whose Cox ring is the polynomial ring $\mathbb{C}[T_1, \dots, T_{r_2}]$, where $r_2 = r_1 + s + 1$, graded by the group $K_2 := K_1 \oplus \mathbb{Z}$. Let $Y'_2 \subseteq Z_2$ be the strict transform of the variety Y_1 as shown in the following diagram

$$\begin{array}{ccc} Y'_2 & \longrightarrow & Z_2 \\ \downarrow & & \downarrow \\ Y_1 & \longrightarrow & Z_1 \longrightarrow W_1. \end{array}$$

Observe that Y'_2 is a blow-up (possibly weighted) of Y_1 at q , whose defining ideal is the following saturation

$$(5.2) \quad I_2 = (\langle T_{r_1+i} T_{r_2}^{d_i} - f_i : 1 \leq i \leq s \rangle + I_1) : \langle T_{r_2} \rangle$$

with respect to the variable T_{r_2} . Let Y_2 be the classical blow-up of Y_1 at q . To conclude that $Y'_2 = Y_2$ and that $\mathbb{C}[T_1, \dots, T_{r_2}]/I_2$ is isomorphic to the Cox ring of

Y_2 , we need to check the following inequality

$$(5.3) \quad \dim I_2 + \langle T_{r_2} \rangle > \dim I_2 + \langle T_{r_2}, T_\nu \rangle,$$

where T_ν is the product of all the T_i 's, for $1 \leq i \leq r_1$, such that $V(T_i)$ does not vanish identically at $p^{-1}(q)$.

5.1. Degree one and two. In this subsection, we provide a presentation for the Cox rings of the del Pezzo elliptic varieties of degree at most two with finite Mordell-Weil group. Our main result is the following.

Theorem 5.1. *Let $\pi: X \rightarrow \mathbb{P}^{n-1}$ be a del Pezzo elliptic fibration of degree $d \leq 2$ having finite Mordell-Weil group. Then the Cox ring of X and its grading matrix are listed in the following table:*

Type	Cox ring	Grading matrix
X_1	$\frac{\mathbb{C}[T_1, \dots, T_{n+2}, S]}{\langle T_1^2 - T_2^3 + T_2 \tilde{f}_4 S^4 + \tilde{f}_6 S^6 \rangle}$ $\tilde{f}_d := f_d(T_1, T_2, T_3 S, \dots, T_{n+2} S)$	$\begin{bmatrix} 3 & 2 & 1 & \dots & 1 & 0 \\ 0 & 0 & -1 & \dots & -1 & 1 \end{bmatrix}$
X_{SS}	$\frac{\mathbb{C}[T_1, \dots, T_{n+3}, S_1, S_2]}{\langle T_{n+2} S_1^4 - T_{n+3} S_2^4 + 2\tilde{h}, T_{n+2} T_{n+3} - \tilde{g} \rangle}$ $\tilde{h} := h(T_1 S_1 S_2, \dots, T_n S_1 S_2, T_{n+1}),$ $\tilde{g} := g(T_1, \dots, T_n)$	$\begin{bmatrix} 1 & \dots & 1 & 1 & 2 & 2 & 0 & 0 \\ -1 & \dots & -1 & 0 & -4 & 0 & 1 & 0 \\ -1 & \dots & -1 & 0 & 0 & -4 & 0 & 1 \end{bmatrix}$
X_2	$\frac{\mathbb{C}[T_1, \dots, T_{n+2}, S_1, S_2]}{\langle T_{n+2}^2 - S_2^2 \tilde{f} - T_n T_{n+1}^3 \rangle}$ $\tilde{f} := \frac{f(T_1 S_1 S_2^2, \dots, T_{n-1} S_1 S_2^2, T_n S_1^2 S_2^2, T_{n+1})}{S_1^2 S_2^2}$	$\begin{bmatrix} 1 & \dots & 1 & 1 & 1 & 2 & 0 & 0 \\ -1 & \dots & -1 & -2 & 0 & -1 & 1 & 0 \\ -1 & \dots & -1 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$

Proof. In order to prove the case X_1 , let Y_1 be the del Pezzo variety given by the polynomial (2.1) and let $q \in Y_1$ be the point of coordinates $(1, 1, 0, \dots, 0)$. The ring R_1 equals $\mathbb{C}[T_1, \dots, T_{n+2}]/I_1$, where I_1 is the principal ideal generated by the polynomial (2.1). We take $I, J \subseteq R_1$ as before and we choose the following homogenous elements f_1, \dots, f_n :

$$T_3, \dots, T_{n+2} \in I$$

as in (5.1), i.e. all of them have $d_i = 1$. Observe that the saturated ideal (5.2) is

$$I_2 = \langle T_{n+2+i} T_{2n+5} - f_i : 1 \leq i \leq n \rangle + I_1$$

since, after applying the substitution $T_{2+i} = T_{n+2+i} T_{2n+5}$ for each $i = 1, \dots, n+2$, the resulting polynomial $T_1^2 - T_2^3 + T_2 \tilde{f}_4 T_{2n+5}^4 + \tilde{f}_6 T_{2n+5}^6$ is not divisible by T_{2n+5} . Finally, according to (5.3), we need to check that

$$\dim I_2 + \langle T_{2n+5} \rangle > \dim I_2 + \langle T_{2n+5}, T_1 T_2 \rangle,$$

and this is easily checked to hold, being I_1 a principal ideal. We conclude that the ring $\mathbb{C}[T_1, \dots, T_{2n+5}]/I_2$ is isomorphic to the the Cox ring of the blow-up X_1 of Y at q . After eliminating the fake linear relations and renaming the variables, we get the claimed presentation for the Cox ring.

We now prove the case X_{SS} . Let Y_1 be the del Pezzo variety given by the polynomial (2.3) and let $q \in Y_1$ be the point of coordinates $(0, \dots, 0, 1, 1)$. The ring R_1 equals $\mathbb{C}[T_1, \dots, T_{n+2}]/I_1$, where I_1 is the principal ideal generated by the polynomial (2.3). We take $I, J \subseteq R_1$ as before and choose the following homogenous elements f_1, \dots, f_{n+1} :

$$T_1, \dots, T_n \in I, \quad T_{n+2} - h \in (I^4 : J^\infty)$$

as in (5.1), that is the first n sections have $d_i = 1$, while $d_{n+1} = 4$. Observe that the ideal in (5.2) is

$$I_2 = \langle T_{n+2+i}T_{2n+4}^{d_i} - f_i : 1 \leq i \leq n+1 \rangle + \langle T_{2n+3}^2 T_{2n+4}^4 + 2T_{2n+3}h' - g' \rangle,$$

where $h' = h(T_{n+3}T_{2n+4}, \dots, T_{2n+2}T_{2n+4}, T_{n+1})$ and $g' = g(T_{n+3}, \dots, T_{2n+2})$. According to (5.3), we can easily check that the following inequality holds:

$$\dim I_2 + \langle T_{2n+4} \rangle > \dim I_2 + \langle T_{2n+4}, T_{n+1}T_{n+2} \rangle.$$

Thus, after eliminating the fake linear relations from I_2 and renaming the variables, we can conclude that the Cox ring and the grading matrix of the blow-up Y_2 of Y_1 at q are the following

$$R_2 = \frac{\mathbb{C}[T_1, \dots, T_{n+2}, S]}{\langle T_{n+2}^2 S^4 + 2h''T_{n+2} - g'' \rangle} \quad \begin{bmatrix} 1 & \dots & 1 & 1 & 2 & 0 \\ -1 & \dots & -1 & 0 & -4 & 1 \end{bmatrix}$$

where $h'' = h(T_1S, \dots, T_nS, T_{n+1})$ and $g'' = g(T_1, \dots, T_n)$. The irrelevant ideal is $J_2 = \langle T_1, \dots, T_n, T_{n+2} \rangle \cap \langle T_{n+1}, S \rangle$. We now repeat the procedure blowing-up Y_2 at the point q'_2 which lies over $q_2 = (0, \dots, 0, 1, -1) \in Y$. Recall that there is a \mathbb{C}^* -equivariant embedding of total coordinate spaces

$$\overline{Y}_1 \rightarrow \overline{Y}_2 \quad (T_1, \dots, T_{n+2}) \rightarrow (T_1, \dots, T_{n+1}, T_{n+2} - h, 1)$$

which induces the birational map $Y_1 \dashrightarrow Y_2$. The image of q_2 is the point of homogeneous coordinates $q'_2 = (0, \dots, 0, 1, -2, 1)$. We choose the following homogenous elements f_1, \dots, f_{n+2} :

$$T_1, \dots, T_n, 2T_{n+1}^2 + T_{n+2}S^4 \in I, \quad T_{n+2}S^4 + 2h'' \in (I^4 : J^\infty)$$

as in (5.1), that is the first $n+1$ sections have $d_i = 1$, while $d_{n+2} = 4$. The ideal in (5.2) is

$$I_3 = \langle T_{n+3+i}T_{2n+6}^{d_i} - f_i : 1 \leq i \leq n+2 \rangle + \langle T_{n+2}^2 S^4 + 2T_{n+2}h''' - T_{2n+6}^4 g''' \rangle$$

where $h''' = h(T_{n+4}T_{2n+6}S, \dots, T_{2n+3}T_{2n+6}S, T_{n+1})$ and $g''' = g(T_{n+4}, \dots, T_{2n+3})$. After eliminating the fake linear relations from the above ideal and renaming the variables, we get the statement for the case X_{SS} .

Finally, let us prove the case X_2 . Let Y_1 be the del Pezzo variety given by the polynomial (2.4) and let $q \in Y_1$ be the point of coordinates $(0, \dots, 0, 1, 0)$. The ring R_1 equals $\mathbb{C}[T_1, \dots, T_{n+2}]/I_1$, where I_1 is the principal ideal generated by the polynomial (2.4). We take $I, J \subseteq R_1$ as before and choose the following homogenous elements f_1, \dots, f_{n+1} :

$$T_1, \dots, T_{n-1}, T_{n+2} \in I, \quad T_n \in (I^2 : J^\infty)$$

as in (5.1), that is the first n sections have $d_i = 1$, while $d_{n+1} = 2$. Observe that the ideal in (5.2) is

$$I_2 = \langle T_{n+2+i}T_{2n+4}^{d_i} - f_i : 1 \leq i \leq n+1 \rangle + \langle T_{2n+2}^2 - f' - T_{2n+3}T_{n+1}^3 \rangle$$

with $f' = T_{2n+4}^{-2}f(T_{n+3}T_{2n+4}, \dots, T_{2n+1}T_{2n+4}, T_nT_{2n+4}^2, T_{n+1})$. According to (5.3) it can be easily checked that

$$\dim I_2 + \langle T_{2n+4} \rangle > \dim I_2 + \langle T_{2n+4}, T_{n+1} \rangle.$$

Thus, after eliminating the fake linear relations from I_2 and renaming the variables, we conclude that the Cox ring and the grading matrix of the blow-up Y_2 of Y_1 at q are the following

$$R_2 = \frac{\mathbb{C}[T_1, \dots, T_{n+3}]}{\langle T_{n+2}^2 - f'' - T_nT_{n+1}^3 \rangle} \quad \begin{bmatrix} 1 & \dots & 1 & 1 & 1 & 2 & 0 \\ -1 & \dots & -1 & -2 & 0 & -1 & 1 \end{bmatrix}$$

with $f'' = T_{n+3}^{-2}f(T_1T_{n+3}, \dots, T_{n-1}T_{n+3}, T_nT_{n+3}^2, T_{n+1})$. The irrelevant ideal of R_2 is $J_2 = \langle T_1, \dots, T_{n-1}, T_{n+2} \rangle \cap \langle T_n, S \rangle$. Now repeat the procedure by blowing up Y_2 at the point $q'_2 = (0, \dots, 0, 1, 1, 1, 0)$ which is the invariant point with respect to the lifted involution $(T_1, \dots, T_{n+3}) \mapsto (T_1, \dots, T_{n+1}, -T_{n+2}, T_{n+3})$, and it corresponds to the generator of the kernel of the differential $d\varphi_q$. We choose the following homogenous elements f_1, \dots, f_n :

$$T_1, \dots, T_{n-1}, T_{n+3} \in I,$$

as in (5.1), i.e. $d_i = 1$ for all the sections. The ideal in (5.2) is

$$I_3 = \langle T_{n+3+i}T_{2n+3}^{d_i} - f_i : 1 \leq i \leq n \rangle + \langle T_{n+2}^2 - T_{2n+3}^2\tilde{f} - T_nT_{n+1}^3 \rangle$$

where $\tilde{f} = T_{2n+3}^{-2}f''(T_{n+4}T_{2n+3}S, \dots, T_{2n+3}T_{2n+3}S, T_{n+1})$. After eliminating the fake linear relations from the above ideal and renaming the variables, we obtain the statement for the case X_2 . \square

5.2. Degree four. In this last subsection, we first provide the following presentation for the Cox rings of the blowing-up of a del Pezzo variety of degree four at a point.

Lemma 5.2. *Let Y be a smooth complete intersection of two quadrics of \mathbb{P}^{n+2} . After possibly applying a linear change of coordinates, the ideal of Y is generated by $x_2x_3 - x_1x_2 + f(x_4, \dots, x_{n+3})$ and $x_2x_3 - x_1x_3 + g(x_4, \dots, x_{n+3})$. The blow-up \tilde{Y} of Y at the point $q = (1, 0, \dots, 0) \in Y$ has the following Cox ring and grading matrix*

$$\frac{\mathbb{C}[T_1, \dots, T_{n+3}, S]}{\langle T_2T_3S^2 - T_1T_2 + f, T_2T_3S^2 - T_1T_3 + g \rangle} \quad \begin{bmatrix} 1 & 1 & 1 & 1 & \dots & 1 & 0 \\ 0 & -2 & -2 & -1 & \dots & -1 & 1 \end{bmatrix}$$

respectively, where $f = f(T_4, \dots, T_{n+3})$ and $g = g(T_4, \dots, T_{n+3})$.

Proof. After applying a linear change of coordinates, we can assume that q is a point of $Y = Q \cap Q'$, where Q is singular at $(1, 1, 0, 0, \dots, 0)$ and Q' is singular at $(1, 0, 1, 0, \dots, 0)$, and that the tangent hyperplanes to Q and Q' at q are $V(x_2)$ and $V(x_3)$, respectively. This proves the first claim.

To prove the second statement, we take R_1 to be $\mathbb{C}[T_1, \dots, T_{n+3}]/I_1$, where I_1 is the ideal of Y , and we apply [4, Algorithm 5.4]. We take $I, J \subseteq R_1$ as before and choose the following homogenous elements f_1, \dots, f_{n+2} :

$$T_4, \dots, T_{n+3} \in I, \quad T_2, T_3 \in (I^2 : J^\infty)$$

as in (5.1), that is the first n sections have $d_i = 1$, while $d_{n+1} = d_{n+2} = 2$. The ideal in (5.2) is

$$I_2 = \langle T_{n+3+i}T_{2n+6}^{d_i} - f_i : 1 \leq i \leq n+2 \rangle + \langle T_{2n+4}T_{2n+5}T_{2n+6}^2 - T_1T_{2n+4} + \tilde{f}, T_{2n+4}T_{2n+5}T_{2n+6}^2 - T_1T_{2n+5} + \tilde{g} \rangle,$$

where $\tilde{f} := f(T_{n+4}, \dots, T_{2n+3})$ and $\tilde{g} := g(T_{n+4}, \dots, T_{2n+3})$. According to (5.3) it can be easily checked that

$$\dim I_2 + \langle T_{2n+6} \rangle > \dim I_2 + \langle T_{2n+6}, T_1 \rangle.$$

After eliminating the fake linear relations from I_2 and renaming the variables, we get the second statement. \square

We conclude with two examples of the computation of the Cox rings for the del Pezzo elliptic varieties of degree four and dimension three. We only report the final results, since the computations have been done with the same procedure as before.

Case X_{43} :

Equations:

$$\begin{aligned} x_2^2 - x_3^2 + x_4^2 + x_5^2 + x_6^2, \\ x_1^2 - x_3^2 + 2x_4^2 + 3x_5^2 + 4x_6^2 \end{aligned}$$

Cox ring $\mathbb{C}[T_1, \dots, T_{13}]/I$, where I is generated by:

$$\begin{aligned} T_1^2 - T_3^2 + 2T_5T_8 - T_6T_7, \\ T_2^2 + 2T_3^2 - T_5T_8 + T_6T_7, \\ T_4T_9 - T_5T_8 - T_6T_7, \\ T_4T_{10}^2 - T_7T_{12}^2 + T_8T_{13}^2, \\ T_4T_{11}^2 - T_5T_{12}^2 - T_6T_{13}^2, \\ T_5T_8T_{11}^2T_{12}^2T_{13}^2 - T_5T_9T_{12}^4T_{13}^2 + T_6T_7T_{11}^2T_{12}^2T_{13}^2 - T_6T_9T_{12}^2T_{13}^4, \\ T_5T_{10}^2 - T_7T_{11}^2 + T_9T_{13}^2, \\ T_6T_{10}^2 + T_8T_{11}^2 - T_9T_{12}^2 \end{aligned}$$

Degree matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -2 & -2 & -2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & -1 & -2 & 0 & 0 & -2 & -2 & 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & -1 & 0 & -2 & 0 & -2 & 0 & -2 & 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & 0 & 0 & -2 & 0 & -2 & -2 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Case X_{22} :

Equations:

$$\begin{aligned} x_2^2 - x_3^2 + x_4^2 + x_5^2 + x_6^2, \\ x_1^2 - x_3^2 + \frac{3}{4}x_4^2 + 2x_5^2 + 3x_6^2 \end{aligned}$$

Cox ring $\mathbb{C}[T_1, \dots, T_{10}]/I$, where I is generated by:

$$\begin{aligned} 2T_1^2 - T_3^2T_8^2T_{10}^2 + 11T_4^2T_7^2T_8^2T_{10}^2 - 8T_4T_5T_7^2T_8^4 + 8T_4T_6T_9^2T_{10}^4 - 4T_5T_6T_8^2T_{10}^2, \\ 2T_2^2 + 3T_3^2 + 3T_4^2T_7^2T_9^2 + 4T_5T_6 \end{aligned}$$

Degree matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -2 & -2 & 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & -1 & -2 & 0 & -2 & 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & -2 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & -2 & 0 & 0 & -1 & 1 \end{bmatrix}$$

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